Mathematics: The core course for A-level; Bostock and Chandler

Chapter 9: Integration

Exercise 9i

(1) Let $I = \int \frac{\cos x}{3 + \sin x}\, dx$

Let $u = 3 + \sin x$, $\therefore du = \cos x \, dx$

So $I = \int \frac{1}{u}\, du = \ln|u| + c$

$= \ln|3 + \sin x| + c$

(2) Let $I = \int \frac{e^x}{1 - e^x}\, dx$

Let $u = 1 - e^x$, $\therefore du = -e^x \, dx$

$\Rightarrow -du = e^x \, dx$

$\therefore I = \int \frac{-1}{u}\, du = -\ln|u| + c$

$= -\ln|1 - e^x| + c$

③ Let $I = \int \frac{2x}{1+x^2}\,dx$

let $u = 1+x^2$, $\therefore du = 2x\,dx$

So $I = \int \frac{1}{u}\,du = \ln|u| + c$

$= \ln|1+x^2| + c$

④ Let $I = \int \frac{\sec^2 x}{1-3\tan x}\,dx$

let $u = 1-3\tan x$, $\therefore du = -3\sec^2 x\,dx$

$\Rightarrow -\frac{1}{3}du = \sec^2 x\,dx$

So $I = -\frac{1}{3}\int \frac{1}{u}\,du = -\frac{1}{3}\ln|u| + c$

$= -\frac{1}{3}\ln|1-3\tan x| + c$

⑤ $I = \int \frac{1}{1+e^x}\,dx = \int \frac{e^{-x}}{e^{-x}+1}\,dx$

let $u = e^{-x}+1$, $\therefore du = -e^{-x}\,dx \Rightarrow -du = e^{-x}\,dx$

$\therefore I = -\int \frac{1}{u}\,du = -\ln|u| + c$

$= -\ln|e^{-x}+1| + c$

(6) Let $I = \int \frac{x^3}{1+x^4}\, dx$

Let $u = 1+x^4$, $\therefore du = 4x^3\, dx$

$\Rightarrow \frac{1}{4} du = x^3\, dx$

So $I = \frac{1}{4}\int \frac{1}{u}\, du = \frac{1}{4}\ln|u| + c$

$= \frac{1}{4}\ln|1+x^4| + c$

(7) Let $I = \int \frac{\cos 2x}{4\sin x \cos x + 1}\, dx = \int \frac{\cos 2x}{2\sin 2x + 1}\, dx$

Let $u = 2\sin 2x + 1$, $\therefore du = 4\cos 2x\, dx$

$\Rightarrow \frac{1}{4} du = \cos 2x\, dx$

So $I = \frac{1}{4}\int \frac{1}{u}\, du = \frac{1}{4}\ln|u| + c$

$= \frac{1}{4}\ln|2\sin 2x + 1| + c$

(8) Let $I = \int \frac{\tan x}{1+\cos x}\, dx = \int \frac{\tan x \cdot \sec x}{\sec x(1+\cos x)}\, dx$

$= \int \frac{\tan x \cdot \sec x}{\sec x + 1}\, dx$

Let $u = \sec x + 1$, $\therefore du = \sec x \tan x \, dx$

So $I = \int \frac{1}{u} \, du = \ln |u| + c$

$= \ln |\sec x + 1| + c$

(9) Let $I = \int \frac{1}{x.\ln x} \, dx$

Let $u = \ln x$, $\therefore du = \frac{1}{x} \, dx$

So $I = \int \frac{1}{u} \, du = \ln |u| + c$

$= \ln |\ln|x|| + c$

(10) Let $I = \int \frac{2x + 3}{x^2 + 3x + 4} \, dx$

Let $u = x^2 + 3x + 4$, $\therefore du = (2x + 3) \, dx$

So $I = \int \frac{1}{u} \, du = \ln |u| + c$

$= \ln |x^2 + 3x + 4| + c$

(11) Let $I = \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx$

Now let $u = \sin x$, $\therefore du = \cos x \, dx$

So $I = \int \frac{1}{u} \, du = \ln |u| + c = \ln |\sin x| + c$

(12) let $I = \int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} \, dx$

let $u = \sec x + \tan x$, $\therefore du = (\sec x \tan x + \sec^2 x) \, dx$

$= \sec x (\tan x + \sec x) \, dx$

So $I = \int \frac{1}{u} \, du = \ln|u| + c$

$= \ln|\sec x + \tan x| + c$

(13) let $I = \int \operatorname{cosec} x \, dx = \int \frac{\operatorname{cosec} x (\operatorname{cosec} x + \cot x)}{\operatorname{cosec} x + \cot x} \, dx$

let $u = \operatorname{cosec} x + \cot x$, $\therefore du = (-\cot x \operatorname{cosec} x - \operatorname{cosec}^2 x) \, dx$

So $-du = \operatorname{cosec} x (\cot x + \operatorname{cosec} x) \, dx$

So $I = -\int \frac{1}{u} \, du = -\ln|u| + c$

$= -\ln|\operatorname{cosec} x + \cot x| + c$

(14) let $I = \int \frac{x}{\sqrt{x^2+1}} \, dx$

let $u = x^2 + 1$, $\therefore du = 2x \, dx \Rightarrow \frac{1}{2} du = x \, dx$

So $I = \frac{1}{2} \int \frac{1}{\sqrt{u}} \, du = \frac{1}{2} \int u^{-1/2} \, du = u^{1/2} + c$

$= (x^2+1)^{1/2} + c$

(15) let $I = \int \frac{\cos x}{\sin^6 x}\, dx$

let $u = \sin x$, so $du = \cos x\, dx$

$\therefore \quad I = \int \frac{1}{u^6}\, du = \int u^{-6}\, du$

$= -\frac{1}{5} u^{-5} + c$

$= -\frac{1}{5} \frac{1}{\sin^5 x} + c$

(16) Let $I = \int \frac{\cos x}{\sqrt{1+\sin x}}\, dx$

let $u = 1+\sin x$, $\therefore du = \cos x\, dx$

$\therefore \quad I = \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du$

$= 2u^{1/2} + c$

$= 2\sqrt{1+\sin x} + c$

(17) let $I = \int \frac{e^x}{(e^x+4)^2}\, dx$

let $u = e^x + 4$, $\therefore du = e^x\, dx$

So $I = \int \frac{1}{u^2}\, du = -\frac{1}{u} + c$

$= -\frac{1}{e^x+4} + c$

(18) let $I = \int \frac{\sec^2 x}{\tan^3 x} dx$

let $u = \tan x$, $\therefore du = \sec^2 x \, dx$

So $I = \int \frac{1}{u^3} du = -\frac{1}{2} u^{-2} + c$

$= \frac{-1}{2} \frac{1}{\tan^2 x} + c$

(19) let $I = \int \frac{\sin x}{\cos^n x} dx$

let $u = \cos x$, $\therefore du = -\sin x \, dx$

So $-du = \sin x \, dx$

$\therefore I = -\int \frac{1}{u^n} du = -\frac{1}{-n+1} u^{-n+1} + c$

$= \frac{1}{n-1} \frac{1}{\cos^{n-1} x} + c$

(20) let $I = \int \frac{\cos x}{\sin^n x} dx$

let $u = \sin x$, $\therefore du = \cos x \, dx$

So $I = \int \frac{1}{u^n} du = \int u^{-n} du = + \frac{1}{-n+1} \cdot u^{-n+1} + c$

$= \frac{-1}{n-1} \cdot \frac{1}{\sin^{n-1} x} + c$

(21) let $I = \int \frac{e^x}{\sqrt{1+e^x}}\, dx$

let $u = 1+e^x$, $\therefore$ $du = e^x\, dx$

so $I = \int \frac{1}{\sqrt{u}}\, du = \int u^{-1/2}\, du$

$= 2 \cdot u^{1/2} + c$

$= 2\sqrt{1+e^x} + c$

(22) let $I = \int \frac{\sec x \cdot \tan x}{3 - \sec x}\, dx$

let $u = (3 - \sec x)\, dx$, $\therefore$ $du = -\sec x \tan x\, dx$

so $-du = \sec x \tan x\, dx$

$\therefore$ $I = -\int \frac{1}{u}\, du = -\ln|u| + c$

$= -\ln|3 - \sec x| + c$

(23) let $I = \int \frac{\operatorname{cosec}^2 x}{(2+\cot x)^4}\, dx$

let $u = 2+\cot x$, $\therefore$ $du = -\operatorname{cosec}^2 x\, dx \Rightarrow -du = \operatorname{cosec}^2 x\, dx$

$\therefore$ $I = \int \frac{1}{u^4}\, du = -\frac{1}{3}\frac{1}{u^3} + c$

$= -\frac{1}{3}\frac{1}{(2+\cot x)^3} + c$

(24) Let $I = \int \frac{x-1}{3x^2-6x+1}\,dx$

let $u = (3x^2-6x+1)\,dx$, $\therefore du = (6x-6)\,dx$

$\Rightarrow \frac{1}{6}\,du = (x-1)\,dx$

$\therefore \quad I = \frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|u| + c$

$= \frac{1}{6}\ln|3x^2-6x+1| + c$

(25) let $I = \int \sin^{-1}x\,dx = \int 1.\sin^{-1}x\,dx$

let $u_1 = \sin^{-1}x$, $\therefore \frac{du_1}{dx} = \frac{1}{\sqrt{1-x^2}}$

& $\frac{dv_1}{dx} = 1$, $\therefore \quad v_1 = x$

So $I = x.\sin^{-1}x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Now let $u_2 = 1-x^2$, $\therefore \quad du_2 = -2x\,dx$

$\Rightarrow -\frac{1}{2}\,du_2 = x\,dx$

So $I = x.\sin^{-1}x - \left[\int -\frac{1}{2}\cdot\frac{1}{\sqrt{u_2}}\,du_2\right]$

$= x.\sin^{-1}x + \frac{1}{2}\int u_2^{-1/2}\,du_2$

$\therefore \quad I = x \cdot \sin^{-1} x + u_2^{1/2} + c$

$= x \cdot \sin^{-1} x + (1 - x^2)^{1/2} + c$

(26) Let $I = \int \tan^{-1} x \, dx$

$= \int 1 \cdot \tan^{-1} x \, dx$

Let $u_1 = \tan^{-1} x$, $\therefore \dfrac{du_1}{dx} = \dfrac{1}{1+x^2}$

and $\dfrac{dv_1}{dx} = 1$, $\therefore v_1 = x$

So $I = x \cdot \tan^{-1} x - \int \dfrac{x}{1+x^2} \, dx$

Now let $u_2 = 1 + x^2$, $\therefore du_2 = 2x \, dx$

$\Rightarrow \frac{1}{2} du_2 = x \, dx$

So $I = x \cdot \tan^{-1} x - \frac{1}{2} \int \dfrac{1}{u_2} \, du_2$

$= x \cdot \tan^{-1} x - \frac{1}{2} \ln |u_2| + c$

$= x \cdot \tan^{-1} x - \frac{1}{2} \ln |1 + x^2| + c.$

(27) let $I = \int \cos^{-1} x \, dx = \int 1 \cdot \cos^{-1} x \, dx$

let $u_1 = \cos^{-1} x$, $\therefore \frac{du_1}{dx} = -\frac{1}{\sqrt{1-x^2}}$

& $\frac{dv_1}{dx} = 1$, $\therefore v_1 = x$

So $I = x \cdot \cos^{-1} x - \int \frac{-1}{\sqrt{1-x^2}} \cdot x \, dx$

$= x \cdot \cos^{-1} x + \int \frac{x}{\sqrt{1-x^2}} \, dx$

now let $u_2 = 1 - x^2$, $\therefore du_2 = -2x \, dx$

$\Rightarrow -\frac{1}{2} du_2 = x \, dx$

So $I = x \cdot \cos^{-1} x + \int -\frac{1}{2} \frac{1}{\sqrt{u_2}} \, du_2$

$= x \cdot \cos^{-1} x - \frac{1}{2} \int u^{-1/2} \, du_2$

$= x \cdot \cos^{-1} x - u^{1/2} + c$

$= x \cdot \cos^{-1} x - (1-x^2)^{1/2} + c$

(28) Let $I = \int \frac{x}{x+1}\,dx$

Can't do Partial fractions here since denominator is already a single linear factor.
Do long division:

$$\begin{array}{r} 1 \\ x+1\,\overline{)\,x} \\ x+1 \\ \hline -1 \end{array}$$

So $I = \int 1 - \frac{1}{x+1}\,dx$

$= x - \ln|x+1| + c$

(29) Let $I = \int \frac{x^2-2}{x^2-1}\,dx$

Do long division 1st :

$$\begin{array}{r} 1 \\ x^2-1\,\overline{)\,x^2-2} \\ x^2-1 \\ \hline -1 \end{array}$$

So $I = \int 1 - \frac{1}{x^2-1}\,dx$

Now $\frac{1}{x^2-1} = \frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \Rightarrow 1 = A(x+1) + B(x-1)$

Let $x = 1$: $1 = 2A \Rightarrow A = \frac{1}{2}$; Let $x = -1$: $1 = -2B \Rightarrow B = -\frac{1}{2}$

So $I = \int 1 - \frac{1/2}{x-1} + \frac{1/2}{x+1}\,dx = x - \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + c$

$= x + \ln\sqrt{\frac{x+1}{x-1}} + c$.

(30) let $I = \int \frac{x^2}{(x+1)(x+2)}\, dx$

can't do partial fractions yet since Numerator & denominator are of same power.
So use long division 1st:

$$\begin{array}{r|l} & 1 \\ \hline x^2+3x+1 & x^2 \\ & x^2+3x+1 \\ \hline & -3x-1 \end{array}$$

So $I = \int 1 - \frac{3x+1}{x^2+3x+1}\, dx$

So now do Partial fractions:

$$\frac{3x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$\therefore \quad 3x+1 = A(x+2) + B(x+1)$$

let $x=-1$: $-2 = A$

$x=-2$: $-5 = -B \Rightarrow B = 5$

So $I = \int 1 + \frac{2}{x+1} - \frac{5}{x+2}\, dx$

$$= x + 2\ln|x+1| - 5\ln|x+2| + C$$

$$= x + \ln\left|\frac{(x+1)^2}{(x+2)^5}\right| + C$$

(Book Ans is wrong)

(31) let $I = \int \frac{x+4}{x}\, dx = \int 1 + \frac{4}{x}\, dx = x + 4\ln|x| + c$

$= x + \ln K|x^4|$

No Partial fractions needed here

(32) $I = \int \frac{x+4}{x+1}\, dx$

Simplify to a proper fraction 1st :

$$\begin{array}{r|l} & 1 \\ \hline x+1 & x+4 \\ & x+1 \\ \hline & 3 \end{array}$$

So $I = \int 1 + \frac{3}{x+1}\, dx$

$= x + 3\ln|x+1| + c$

$= x + \ln K|(x+1)^3|$, where $K = \ln c$

(33) let $I = \int \frac{2x}{(x-2)(x+2)}\, dx$

let $\frac{2x}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$

So $2x = A(x+2) + B(x-2)$

let $x = 2$: $4 = 4A \Rightarrow A = 1$

$x = -2$: $-4 = -4B \Rightarrow B = 1$

$$\therefore \; I = \int \frac{1}{x-2} + \frac{1}{x+2} \, dx$$

$$= \ln|x-2| + \ln|x+2| + c$$

$$= \ln K.(x-2).(x+2)$$

$$= \ln K(x^2-4)$$

34 Let $I = \int \frac{3u+4}{u(u+1)} \, du$

Let $\frac{3u+4}{u(u+1)} = \frac{A}{u} + \frac{B}{u+1}$

So $3u+4 = A(u+1) + Bu$

Let $u=0$: $4 = A$

$u=-1$: $1 = -B \Rightarrow B = -1$

So $I = \int \frac{4}{u} - \frac{1}{u+1} \, du$

$$= 4\ln|u| - \ln|u+1| + c$$

$$= \ln\left|\frac{u^4}{u+1}\right| + C$$

35 Let $I = \int \frac{x^2 + x + 5}{x(x+1)}\, dx$

First, convert to a proper fraction. So by long division

$$\begin{array}{r|l} & 1 \\ x^2 + x & x^2 + x + 5 \\ & -(x^2 + x) \\ \hline & 5 \end{array}$$

$\therefore$

$$I = \int 1 + \frac{5}{x(x+1)}\, dx$$

let $\frac{5}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$

$\therefore \quad 5 = A(x+1) + Bx$

let $x = 0: \quad 5 = A$

$x = -1: \quad 5 = -B \implies B = -5$

So $I = \int 1 + \frac{5}{x} - \frac{5}{x+1}\, dx$

$$= x + 5\ln|x| - 5\ln|x+1| + c$$

$$= x + \ln\left|\left(\frac{x}{x+1}\right)^5\right| + c$$

(36) let $I = \int \frac{3-y}{(y-1)(y-2)}\, dy$

let $\frac{3-y}{(y-1)(y-2)} = \frac{A}{y-1} + \frac{B}{y-2}$

$\therefore \quad 3-y = A(y-2) + B(y-1)$

if $y=1: \quad 2 = -A \Rightarrow A = -2$

$y=2: \quad 1 = B$

So $I = \int -\frac{2}{y-1} + \frac{1}{y-2}\, dy$

$= -2\ln|y-1| + \ln|y-2| + c$

$= \ln\left|\frac{y-2}{(y-1)^2}\right| + c$

(37) let $I = \int \frac{2z-5}{z^2-5z+6}\, dz$

let $\frac{2z-5}{z^2-5z+6} = \frac{A}{z-3} + \frac{B}{z-2}$

So $\quad 2z-5 = A(z-2) + B(z-3)$

if $z=2: \quad -1 = -B \Rightarrow B=1$

$z=3: \quad 1 = A$

$$\therefore \quad I = \int \frac{1}{z-3} + \frac{1}{z-2}\, dz$$

$$= \ln|z-3| + \ln|z-2| + c$$

$$= \ln K\left|(z-2)(z-3)\right|$$

(38) $I = \displaystyle\int \frac{12x}{(2-x)(3-x)(4-x)}\, dx$

let $\dfrac{12x}{(2-x)(3-x)(4-x)} = \dfrac{A}{2-x} + \dfrac{B}{3-x} + \dfrac{C}{4-x}$

So $\quad 12x = A(3-x)(4-x) + B(2-x)(4-x) + C(2-x)(3-x)$

If $x = 3: \quad 36 = -B \Rightarrow B = -36$

$x = 4: \quad 48 = 2C \Rightarrow C = 24$

$x = 2: \quad 24 = 2A \Rightarrow A = 12$

So $\quad I = \displaystyle\int \frac{12}{2-x} - \frac{36}{3-x} + \frac{24}{4-x}\, dx$

$$= -12\ln|2-x| + 36\ln|3-x| - 24\ln|4-x| + c$$

$$= \ln\left|\frac{(3-x)^{36}}{(2-x)^{12}(4-x)^{24}}\right| + c$$

39 $I = \int \frac{x^2+2x+4}{(2x-1)(x^2-1)}\,dx$

$= \int \frac{x^2+2x+4}{(2x-1)(x-1)(x+1)}\,dx$

Let $\frac{x^2+2x+4}{(2x-1)(x-1)(x+1)} = \frac{A}{2x-1} + \frac{B}{x-1} + \frac{C}{x+1}$

Then $x^2+2x+4 = A(x-1)(x+1) + B(2x-1)(x+1)$
$+ C(2x-1)(x-1)$

If $x=1$: $7 = 2B \Rightarrow B = 7/2$
$x=-1$: $3 = 6C \Rightarrow C = \frac{1}{2}$
$x=1/2$: $5\frac{1}{4} = -\frac{3}{4}A \Rightarrow A = -7$

So

$I = \int \frac{-7}{2x-1} + \frac{7/2}{x-1} + \frac{1/2}{x+1}\,dx$

$= -\frac{7}{2}\ln|2x-1| + \frac{7}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + c$

$= \ln\left|\frac{\sqrt{(x+1)(x-1)^7}}{\sqrt{(2x-1)^7}}\right| + c$

(40) let $I = \int \frac{4u^2+3u-2}{(u+1)(2u+3)}\,du$

Convert to a proper fraction:

$$\begin{array}{r} 2 \\ 2u^2+5u+3\;\big)\;4u^2+3u-2 \\ -\;\underline{4u^2+10u+6} \\ -7u-8 \end{array}$$

So $I = \int 2 - \frac{7u+8}{(u+1)(2u+3)}\,du$

let $\frac{7u+8}{(u+1)(2u+3)} = \frac{A}{u+1} + \frac{B}{2u+3}$

$\therefore \quad 7u+8 = A(2u+3) + B(u+1)$

if $u=-1: \quad 1 = A$

$u = -\frac{3}{2}: \quad -\frac{5}{2} = -\frac{1}{2}B \quad \Rightarrow \quad B=5$

So $I = \int 2 - \frac{1}{u+1} - \frac{5}{2u+3}\,du$

$= 2u - \ln|u+1| - \frac{5}{2}\ln|2u+3| + c$

$= 2u + \ln\left|\frac{1}{(u+1)\sqrt{(2u+3)^5}}\right| + c$